A NOTE ON SINGULARITIES OF THE 3-D EULER EQUATION

S. Tanveer*
Mathematics Department
Ohio State University
Columbus, OH 43210

ABSTRACT

In this paper, we consider analytic initial conditions with finite energy, whose complex spatial continuation is a superposition of a smooth background flow and a singular field. Through explicit calculation in the complex plane, we show that under some assumptions, the solution to the 3-D Euler equation ceases to be analytic in the real domain in finite time.

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1. INTRODUCTION

One fundamental question in fluid mechanics is whether there exists a velocity field $\vec{u}(\vec{x}, t)$ and a corresponding vorticity field $\vec{\omega}(\vec{x}, t)$ satisfying the incompressible 3-D Euler equations

$$\vec{\omega}_t \ + \ \vec{u} \cdot \nabla \vec{\omega} \ = \ \vec{\omega} \cdot \nabla \ \vec{u} \tag{1}$$

$$\nabla \cdot \vec{u} \ = \ 0 \tag{2}$$

$$\nabla \ \times \ \vec{u} \ = \ \vec{\omega} \tag{3}$$

so that for some time $t \ > \ 0$ these fields are singular when $\vec{u}(\vec{x}, 0)$ and $\vec{\omega}(\vec{x}, 0)$ are initially smooth and the energy is finite, i.e.

$$\int d^3 \ \vec{x} \ |\vec{u}(\vec{x}, \ 0) \ |^2 \ < \ \infty \tag{4}$$

The answer to this and related question of enstrophy blow up in the limit of infinite Reynolds number have important ramifications in turbulence[1].

There have been many investigations in the literature to address this question. Exact solutions are known[2] for which a smooth flow becomes singular in finite time; the velocity field in all such cases do not vanish at infinity and therefore the energy constraint (4) is violated. This constraint is important since Childress et al[3] showed that without the energy constraint, there are explicit singular solutions of the 2-D Euler equations which is impossible[4,5,6] for flows with finite energy. There have been also been a number of mathematically rigorous work[7,8] on this issue. While there are no definitive theorems one way or the other, a number of necessary conditions for finite time singularities have been proved. There have also been quite a number of numerical investigations[9−12] as well. Nonetheless, a definitive conclusion based on these is difficult to reach. Motivated by physical insight in the analogous magnetohydrodynamics equations, arguments based on an assumed multi-scale expansion were presented[13] for the occurrence of finite time singularity for flows with bounded velocity at infinity, rather than bounded energy.

Recently, there have been investigations involving singularities in the complex spatial domain. Such studies become relevant to the issue at hand if complex singularities are found to impinge the real physical domain later in time. Caflisch[14] simplifies the 3-D Euler equation through a so called "Moore approximation". Within this approximation, a finite time singularity is predicted. However, the relevance of this calculation to the actual 3-D Euler equation is unclear since the initial data in the real domain is complex and the initial value problem is locally ill-posed unlike the actual 3-D Euler equations. Tanveer & Speziale[15] recently looked at a special class of initial condition that are analytic and real in the real x_1, x_2, x_3 domain ($\vec{x} = (x_1, x_2, x_3)$) with a complex spatial plane continuation (i.e. complex x_1,

x_2, x_3) that includes singularities on a set of points $\vec{x}$ that are zeros of a a scalar function of $\vec{x}$. At such points, the analytically extended velocity is finite but vorticity infinite. The initial condition is decomposed into a smooth spatial flow and a singular part. Under certain assumptions on existence of solutions in this extended complex domain, it was shown that this decomposition remains invariant with time, though initial singularities become advected with the smooth part of the velocity field (call it the background flow). To the extent that these assumptions hold for any class of initial conditions, the results posed no additional restriction on the choice of the background flow other than that it is spatially analytic in a neighborhood of complex points where the remaining part of the velocity and vorticity fields are singular. Indeed for steady background flows, it was seen that the dynamics of the complex singularities resulted in sufficient conditions for instability through analysis of three ordinary differential equations. Interestingly enough, these differential equations are identical to those found by Lifschitz & Hameiri[16] and Vishik & Friedlander[17], who investigated instability of 3-D Euler flows using a WKB analysis for short wavelength disturbances. The latter results have also been established rigorously.

Recently, aside from presenting arguments for finite time singularity for a class of flow with bounded velocity but not energy, Bhattacharjee et al[18] applied the complex singularity approach of Tanveer & Speziale[15] to a background flow that is spatially smooth in the complex plane, but temporally singular. They presented arguments suggesting that if the the assumptions made by Tanveer & Speziale[15] are valid, complex singularities can impinge the real $\vec{x}$ domain in finite time resulting in an additional collapsing length scale besides that of the background flow. However, no information on the dynamic evolution of the singular part of velocity and vorticity is obtained. Further, it is not clear if an initially analytic divergence free velocity field in the real domain satisfying the energy constraint (4) can be decomposed in the complex $\vec{x}$ domain into a spatially smooth background flow that has explicit solution and a singular part.

In this paper, largely motivated by the recent work of Bhattacharjee et al[18], we present analytic initial conditions $\vec{u}(\vec{x}, 0)$ in the unbounded real domain satisfying the energy constraint (4) for which the analyticity strip width of solution to (1)-(3) is found to shrink to zero in a finite time under some clearly stated assumptions. Since the assumptions above have not been verified, the results do not provide a clear answer to the question of finite time singularity for finite energy smooth initial conditions; nonetheless, we suspect that the results will be of value in settling this issue since the assumptions can be checked numerically through appropriately designed calculations.

2. INITIAL CONDITIONS

Consider initial velocity field given by

$$\vec{u}(\vec{x},\ 0) \ = \ \vec{u}_s\ (\vec{x},0) + \ f(d[\vec{x},\ 0])\ \vec{q}(\vec{x},0)\ , \tag{5}$$

where

$$\vec{u}_s\ (\vec{x},0) \ = \ (-\frac{1}{2}x_1 - \frac{\beta}{2}\ x_2,\ \frac{\beta}{2}x_1 - \frac{1}{2}x_2,\ x_3)\ , \tag{6}$$

$$f(d) \ = \ d^{1/2}, \tag{7}$$

$$d(\vec{x},0) \ = \ g(\ r^2)\ \ , \tag{8}$$

where $\vec{x} = (x_1, x_2, x_3)$, $r^2 = x_1^2\ +\ x_2^2\ +\ x_3^2$ and

$$g(s) \ = \ \frac{a^2\ (1+s)}{1\ +\ 2s\ +\ a^2\ (s^2\ +\ s^3\ )}\ , \tag{9}$$

for some small real and positive constant a^2 and

$$\vec{q}(\vec{x},0) \ = \ [\ h(r^2)\ -\ r^2\ ]\ \vec{u}_s(\vec{x},0)\ +\ 2\ h'(r^2)\ \vec{x}\ \times\ \vec{W}\ (\vec{x})\ , \tag{10}$$

where

$$h(r^2) \ = \ 2\frac{g(r^2)}{g'(r^2)}\ +\ r^2\ \ . \tag{11}$$

$$\vec{W}(\vec{x}) = \left[-\frac{1}{2}x_2x_3\ ,\ \frac{1}{2}\ x_1\ x_3,\ -\ \frac{\beta}{4}\ (x_1^2 + x_2^2)\right]\ , \tag{12}$$

The initial condition (5) corresponds to an axi-symmetric flow with swirl. Nonetheless, we do not make much use of this symmetry since we closely follow the formulation of Tanveer & Speziale[15] that was developed for the general 3-D Euler equation. Now, we list below some claims that are shown to be true.

Claim i. The function $\vec{q}(\vec{x},0)$ is real and analytic everywhere in the real $(x_1,\ x_2,\ x_3)$ domain. It is also analytic in the finite complex $\vec{x}$ plane except at points where $g'(r^2)\ =\ 0$. These points are far away from the real domain for small a^2.

First notice from (9) that

$$g'(r^2) \ = \ -\frac{a^2}{[1+2r^2\ +\ a^2(r^4+r^6)]^2}\ \left[1 + 2a^2r^2 + 4a^2r^4 + 2a^2r^6\right] \tag{13}$$

Notice that the above is nonzero for real $\vec{x}$ for which $r^2\ >\ 0$. When a^2 is small, a zero in the complex $(x_1,\ x_2,\ x_3)$ plane occurs only at points far away from the real domain, i.e. for $|Im\ x_j|$ large, atleast for some j. Further, while there can be poles of $g(r^2)$, $h(r^2)$ defined in (11) is clearly analytic at such points. Thus, the function $h(r^2)$ is analytic except at points where $g'(r^2) = 0$. The same is clearly true for $h'(r^2)$. This establishes claim (i) since $\vec{u}_s\ (\vec{x}, 0)$ and $\vec{W}(\vec{x})$ are obviously analytic functions of $\vec{x}$, i.e. of complex variables x_1, x_2 and x_3.

Claim ii. The initial condition $\vec{u}(\vec{x},\ 0)$ is analytic everywhere in the real $(x_1,\ x_2,\ x_3)$ domain with singularities in the complex plane determined by the relation $g(r^2) = 0$, $g(r^2) = \infty$ and $g'(r^2) = 0$.

We note that $d(\vec{x}, 0) = g(r^2)$ is analytic and nonzero everywhere except at points in the complex plane when the denominator in (9) vanishes. For small a^2, zero of the denominator in (9) occurs near $r^2 = -\frac{1}{2}$, while the other zeros occur far away from the real domain. $d(\vec{x}, 0)$ is also zero in the complex $(x_1,\ x_2,\ x_3)$ plane at points where $r^2 = -1$. Raising d to a fractional power, we obtain singularities of $f(d[\vec{x}, 0])$ at $r^2 = -1$ (where $g(r^2) = 0$) and at other complex points where $g(r^2) = \infty$. From the decomposition in (5) and claim (i) above, claim (ii) follows.

Claim iii. The energy constraint (4) is satisfied.

To show that constraint (4) is satisfied, it suffices to show that for real $\vec{x}$ with $r \rightarrow \infty$, $\vec{u}(\vec{x}, 0) = O(r^{-3})$. From the expression of $g(r^2)$ in (9) and $g'(r^2)$ in (13), it is not difficult to show from (11) that for large r, $h(r^2) = O(r^{-2})$, $h'(r^2) = O(r^{-4})$. Further, from (12), $\vec{W} = O(r^2)$ and therefore from (5), it follows that $\vec{u}(\vec{x}, 0) = O(r^{-3})$. So claim (iii) is established.

Claim iv. The initial condition $\vec{u}(\vec{x}, 0)$ given by (1) satisfies the incompressibility condition. Indeed, each of $\vec{u}_s(\vec{x}, 0)$ and $f(d)\ \vec{q}(\vec{x}, 0)$ is divergence free.

The divergence condition can be shown to be valid directly by calculating the divergence of the right hand side of (5), using the expressions for $f(d[\vec{x},\ 0]$ and $\vec{q}(\vec{x}, 0)$ as shown above.

3. DYNAMICS

Given that the initial data is analytic in the real domain, it would appear reasonable to assume the following, which is listed as

Assumption (i)

The initial value problem (1)-(3) with analytic initial condition (5) has an analytic solution in the real domain for t in some interval $(0,\ t_0)$ for some $t_0 > 0$.

Note: If an analytic solution to (1)-(3) does not exist beyond t_0, then we will have shown that the analyticity width shrinks to zero in finite time. In that case, there is nothing more to demonstrate. So, we will assume otherwise and let $t_0 = \infty$.

We now define $\vec{u}_s(\vec{x},t)$, hence defined to be the background flow, to be

$$\vec{u}_s\ (\vec{x},t)\ =\ \frac{1}{1-t}\left[-\frac{1}{2}x_1 - \frac{\beta}{2}\ x_2,\ \frac{\beta}{2}x_1 - \frac{1}{2}x_2,\ x_3\right]\ , \tag{14}$$

corresponding to which the vorticity $\vec{\omega}_s = \nabla\ \times\ \vec{u}_s$ is

$$\vec{\omega}_s\ (\vec{x},\ t)\ =\ \frac{\beta}{1-t}\ [0,\ 0,\ 1] \tag{15}$$

Notice that (14) and (15) satisfy the Euler equations (1)-(3), and are singular at $t\ =\ 1$, though there are no spatial singularities; however, since it is unbounded at ∞, the energy constraint (4) is not satisfied by $\vec{u}_s$.

We define $d(\vec{x},t)$ to be the solution to

$$d_t\ +\ \vec{u}_s\ \cdot\ \nabla\ d\ =\ 0 \tag{16}$$

satisfying initial condition (8). From the standard method of characterestics,

$$d(\vec{x}(t),t)\ =\ d(\vec{x}(0),0)\ =\ g(x_1^2(0)\ +\ x_2^2(0)\ +\ x_3^2(0)\ ) \tag{17}$$

along trajectories determined from

$$\frac{d\vec{x}}{dt}\ =\ \vec{u}_s\ (\vec{x},\ t) \tag{18}$$

Solving (18),

$$\vec{x}(t) = [x_1(t), x_2(t), x_3(t)] \tag{19}$$

where

$$x_1(t)\ =\ x_1(0)(1-t)^{1/2}\ cos\ \left[\frac{\beta}{2}\ ln(1-t)\right]\ -\ x_2(0)(1-t)^{1/2}\ sin\ \left[\frac{\beta}{2}\ ln(1-t)\right] \tag{20}$$

$$x_2(t)\ =\ x_2(0)(1-t)^{1/2}\ cos\ \left[\frac{\beta}{2}\ ln(1-t)\right]\ +\ x_1(0)(1-t)^{1/2}\ sin\ \left[\frac{\beta}{2}\ ln(1-t)\right] \tag{21}$$

$$x_3(t)\ =\ \frac{x_3(0)}{1-t} \tag{22}$$

From (17)-(22), it follows that

$$d(\vec{x},t)\ =\ g\ \left(\frac{1}{1-t}\ [x_1^2 + x_2^2] + (1-t)^2\ x_3^2]\right) \tag{23}$$

We now decompose velocity $\vec{u}(\vec{x},t)$ and the vorticity $\omega(\vec{x},\ t)\ =\ \nabla\ \times\ \vec{u}(\vec{x},t)$ into

$$\vec{u}(\vec{x},\ t)\ =\ \vec{u}_s(\vec{x},t)\ +\ f\ (d[\vec{x},\ t])\ \vec{q}(\vec{x},\ t) \tag{24}$$

$$\vec{\omega}(\vec{x},t) \;=\; \vec{\omega}_s(\vec{x},\; t) \;+\; f'(d[\vec{x},t])\; \vec{p}(\vec{x},t) \tag{25}$$

where initial condition (5) is satisfied. Then, in order for the vorticity $\vec{\omega}(\vec{x},t)$ and the velocity $\vec{u}(\vec{x},t)$ to satisfy (1)-(3), it is clear on substitution that $\vec{p}(\vec{x},t)$ and $\vec{q}(\vec{x},t)$ must satisfy

$$\vec{p}_t + \vec{u}_s \cdot \nabla\vec{p} - \vec{p}\frac{f^2 f''}{f'^2}\nabla\cdot\vec{q} \;+\; f(\vec{q}\cdot\nabla)\;\vec{p} \;+\; \frac{f}{f'}(\vec{q}\cdot\nabla)\;\vec{\omega}_s = \vec{q}(\vec{\omega}_s\cdot\nabla\; d)$$

$$+\;(\vec{p}\cdot\nabla)\;\vec{u}_s \;+\; \frac{f}{f'}\;(\vec{\omega}_s\cdot\nabla)\;\vec{q} + \vec{q}f(\nabla d)\cdot(\nabla\times\vec{q}) + f\vec{p}\cdot\nabla\vec{q} \tag{26}$$

$$\frac{f}{f'}\nabla\cdot\vec{q} + \vec{q}\cdot\nabla d = 0 \tag{27}$$

$$\frac{f}{f'}\nabla\times\vec{q} + \nabla d\times\vec{q} = \vec{p} \tag{28}$$

The existence of analytic $\vec{p}\;(\vec{x},\; t)$ and $\vec{q}\;(\vec{x},\; t)$ in the real domain, satisfying (26)-(28) for t in the interval $(0,\; 1)$ follows from assumption (i) and the decomposition (24) and (25).

We now introduce another assumption:

Assumption (ii)

$\vec{p}(\vec{x},t)$ and $\vec{q}(\vec{x},t)$ are analytically continuable upto the set of points $\vec{x}_0(t)$, where $d(\vec{x}_0(t),t) \;=\; 0$. Further, each of $\nabla\;\vec{q}$, and $\nabla\vec{p}$ is bounded at $d = 0$.

Remark: Since each of $\frac{f^2 f''}{f'^2}$, f and $\frac{f}{f'}$ are continuous functions of d at $d = 0$, equations (26)-(28) can be seen to have continuous coefficients at $d \;=\; 0$. So, $\vec{p}(\vec{x},t)$ is expected to be continuously differentiable at the set of points $\vec{x}_0(t)$, atleast for early time. Moreover, if $\vec{p}(\vec{x}_0(t),\; t)$ is known apriori to be just a locally analytic function of t, then the results below will hold for $\vec{p}(\vec{x}_0(t),\; t)$ for all t in the interval $(0,\; 1)$, regardless of the assumed boundedness in assumption (ii).

We now define

$$\vec{A}(t) = \nabla d(\vec{x}_0(t),t) \tag{29}$$

$$\vec{B}(t) = \vec{q}(\vec{x}_0(t),t) \tag{30}$$

$$\vec{C}(t) = \vec{p}(\vec{x}_0(t),t) \tag{31}$$

We denote

$$\vec{x}_0(0) \;=\; (\xi,\eta,\zeta)$$

Since $d(\vec{x}_0(0),0) = g(\xi^2+\eta^2+\zeta^2) = 0$, it follows that

$$\xi^2+\eta^2+\zeta^2 \;=\; -1$$

From (23) and (29),

$$\vec{A}(0) \;=\; 2\;g'(-1)\;\;[\;\xi,\;\eta,\;\zeta] \tag{32}$$

Further from (10) and (30), it follows that

$$\vec{B}(0) \;=\; 2\;h'(-1)\;\left[-\;\frac{\beta}{4}\;\eta\;(\xi^2+\eta^2)-\frac{1}{2}\xi\;\zeta^2,\;\;\frac{\beta}{4}\;\xi\;(\xi^2+\eta^2)-\frac{1}{2}\eta\;\zeta^2,\;\;\frac{1}{2}\;\zeta\;(\xi^2+\eta^2)\right] \tag{33}$$

It is clear that since $d(\vec{x}_0(t),t) \;=\; 0$, each of f, f/f' are zero at $d=0$, then with assumption (ii), (27) and (28) imply

$$\vec{A}\cdot\vec{B}=0 \tag{34}$$

$$\vec{A}\times\vec{B}=\vec{C} \tag{35}$$

It follows that

$$\vec{C}\times\vec{A}=(\vec{A}\cdot\vec{A})\vec{B} \tag{36}$$

Also, from (32), (33) and (35), it follows that

$$\begin{aligned}\vec{C}(0) \;=\; 4\;g'(-1)\;h'(-1)\;\Bigg(&-\frac{\beta}{4}\;\xi\;\zeta\;(\xi^2+\eta^2)-\frac{1}{2}\;\eta\;\zeta,\;-\frac{\beta}{4}\;\eta\;\zeta\;(\xi^2+\eta^2)\;+\;\frac{1}{2}\;\xi\;\zeta,\\ &-\frac{\beta}{4}\;\zeta^2\;(\xi^2+\eta^2)\;-\;\frac{\beta}{4}\;(\xi^2+\eta^2)\Bigg)\end{aligned} \tag{37}$$

Furthermore, with assumption (ii), it follows from (26) that

$$\frac{d\vec{C}}{dt}=(\vec{\omega}_s\cdot\vec{A})\vec{B}+\mathbf{T}\vec{C} \tag{38}$$

where $\mathbf{T}$ is a second rank tensor whose elements are defined by

$$T_{jk}=\frac{\partial u_{sj}}{\partial x_k} \tag{39}$$

In our case, the only nonzero elements of $\mathbf{T}$ are

$$T_{11}=-\frac{1}{2(1-t)}=T_{22},\quad T_{33}\;=\;\frac{1}{1-t}\quad,T_{12}=-T_{21}=-\frac{\beta}{2(1-t)} \tag{40}$$

From (38), we get

$$\frac{d\vec{C}}{dt}=\mathbf{S}\vec{C} \tag{41}$$

where $\mathbf{S}$ is a second rank tensor whose elements are given by

$$S_{jk}=T_{jk}+\epsilon_{jkl}A_l\;\frac{\vec{\omega}_s\cdot\vec{A}}{\vec{A}\cdot\vec{A}} \tag{42}$$

given that ϵ_{jkl} is the usual Levi-Civita tensor and the Einstein summation convention on repeated indices has been used.

For $t > 0$, it follows from (23) and (29) that

$$\vec{A}(t) = \frac{2\, g'(-1)}{1-t} \left[x_{1_0}(t),\ x_{2_0}(t),\ (1-t)^3\, x_{3_0}(t) \right] \tag{43}$$

where $\vec{x}_0(t) = [x_{1_0}(t),\ x_{2_0}(t),\ x_{3_0}(t)]$ is determined from (20)-(22) with $\vec{x}_0(0) = (\xi,\ \eta,\ \zeta)$. Using (20) and (21), it follows

$$\vec{A} \cdot \vec{A} = \frac{4 g'^2(-1)}{1-t} \left[\xi^2 + \eta^2 + (1-t)^3\, \zeta^2 \right] \tag{44}$$

Using (43) and (44) in the expression for $\mathbf{S}$ in (42), we find that the components of equation (41) satisfy the following system of ordinary differential equations:

$$\frac{dC_1}{dt} = T_{11}\, C_1 + T_{12}\, C_2 + \frac{\zeta\beta}{\xi^2+\eta^2+(1-t)^3\zeta^2} \left[(1-t)^3 x_{3_0}(t)\, C_2 - x_{2_0}(t)\, C_3 \right] \tag{45}$$

$$\frac{dC_2}{dt} = T_{21}\, C_1 + T_{22}\, C_2 + \frac{\zeta\beta}{\xi^2+\eta^2+(1-t)^3\zeta^2} \left[x_{2_0}(t)\, C_3 - (1-t)^3\, x_{3_0}(t)\, C_1 \right] \tag{46}$$

$$\frac{dC_3}{dt} = T_{33}\, C_3 + \frac{\zeta\beta}{\xi^2+\eta^2+(1-t)^3\zeta^2} \left[x_{2_0}(t)\, C_1 - x_{1_0}(t)\, C_2 \right] \tag{47}$$

Exact solutions to this is possible when $\zeta = 0$ in which case each of the components C_1, C_2 is identically zero initally and remains so for $t > 0$. Using $g'(-1) = -\frac{a^2}{(2a^2-1)^2}$, $h'(-1) = 3$, it follows from (37) and (47) that the third component of $\vec{C}$ is given by

$$C_3(t) = -\frac{3a^2\beta}{(2a^2-1)^2\,(1-t)} \tag{48}$$

Therefore for $\zeta = 0$,

$$\vec{C}(t) = -\frac{3a^2\beta}{(2a^2-1)^2\,(1-t)}\,(0,\ 0,\ 1) \tag{49}$$

It follows from (37), (43) and (44) that in such cases

$$\vec{B}(t) = \frac{3\beta}{2(1-t)^{1/2}}\,[\eta,\ -\xi,\ 0] \tag{50}$$

It is clear from (49) and (50) and the definitions (29)-(31) that each of $\vec{p}(\vec{x}_0(t),t)$ and $\vec{q}(\vec{x}_0(t),t)$ is singular as $t \to 1^-$, at the same time as complex singular points

$$\vec{x}_0(t) = (x_{1_0}(t),\ x_{2_0}(t),\ 0),$$

determined from (20) and (21) approaches (0, 0, 0) as $(1-t)^{1/2}$.

Since the total vorticity is given by (25), it follows that sufficiently close to these complex singular points $\vec{x}_0(t)$,

$$\vec{\omega}(\vec{x},t) \sim \frac{\beta}{1-t}\,[0,\ 0,\ 1] + \frac{1}{2} \left[\vec{A}(t) \cdot (\vec{x} - \vec{x}_0(t)) \right]^{-1/2} \vec{C}(t) \tag{51}$$

where $\vec{A}(t)$ is determined from (43) (with $x_{3_0}(t) = 0$ in this case) and $\vec{C}(t)$ is determined from (49). It is clear from (50) that the singularities of $\vec{\omega}_s$ and $f'(d)\ \vec{p}$ do not cancel out as $t \rightarrow 1^-$ at $\vec{x}_0(t)$.

In the general case, when $\zeta \neq 0$, (45)-(47) do not appear to be exactly solvable. Nonetheless, asymptotic considerations suggest that in such cases, all components of $\vec{C}(t)$ generally become singular as $t \rightarrow 1^-$.

4. IMPLICATIONS OF THE CALCULATIONS

The results (51) in the complex plane together with the result that certain class of initially complex singularities $\vec{x}_0(t)$ impinge (0,0,0) in finite time, would appear to suggest that the initial value problem (1)-(3) with initial condition (5) becomes singular in the real domain in finite time. This would be true if one can assume the following:

Assumption (iii)

If analytic solution to (1)-(3) with initial condition (5) exist for t in (0, 1), then as $t \rightarrow 1^-$, one or more singularities $\vec{x}_0(t)$, whose nonzero components are computed through (21) and (22), is located on the same Riemann sheet as the real physical domain.

Note: If assumption (iii) is correct, then the point (0,0,0) at which $\vec{x}_0(t)$ impinges will actually be part of the physical real domain and not of another Riemann sheet.

In the event that assumption (iii) is valid, then the calculations in section 3 show that that there must be a collapsing physical length scale near the origin, proportional to $(1-t)^{1/2}$, corresponding to the complex singularity distance from the real domain. This is in addition to the length scale associated with $\vec{u}_s(\vec{x}_s, t)$ in (14) that obviously scales as $(1-t)$. This was suggested by Bhattacharjee et al[18].

Another point to note about the superposition (25) is that for a fixed real $\vec{x}$ different from 0, as $t \rightarrow 1^-$, $\vec{\omega}$ need not be singular; temporal singularities in $\vec{\omega}_s$ can cancel out that of $f'(d)\ \vec{p}$, as we suspect it will, though we are unable to demonstrate this convincingly.

The arguments above are not applicable if the solution to (1)-(3) cease to be analytic in the real domain for $t < 1$, as can happen if complex singularities of $\vec{p}$, created in the dynamical process, impinges the real domain for $t < 1$. Note that for small a^2, there exists complex poles of $g(r^2)$ and hence of $d(\vec{x}, 0)$ near $r^2 = -\frac{1}{2}$; from examination of (26)-(27), it is clear that singularities of $\vec{p}$ are expected to be created at such points at $t = 0^+$.

Whatever the case, our calculations above show that if assumptions (i)-(iii) are valid, then the solution to (1)-(3) ceases to be analytic in the real domain beyond $t = 1$. It would be interesting to check these numerically through schemes that monitor the analyticity width as a function of t. Determination of the physical nature of such singularities, if they occur, will also be interesting.

5. CONCLUSION

We have considered an initial condition for the three dimensional Euler equation that is analytic in the real $\vec{x}$ domain and contains finite energy. By making three clearly stated assumptions, we find through explicit calculations that the solution must lose analyticity in finite time. Further study, both analytical and numerical, is likely to shed some light on the validity of the assumptions. If these assumptions are true, numerical calculations is also likely to suggest if the solution loses analyticity before $t = 1$ or if it does so at $t = 1$ in the manner shown explicitly in this paper.

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